Geodesic interpolation inequalities on Heisenberg groups

Inégalités d'interpolation géodésique sur les groupes de Heisenberg

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Abstract

In this Note we present geodesic versions of the Borell-Brascamp-Lieb, Brunn-Minkowski and entropy inequalities on the Heisenberg group $\mathbb{H}^n$. Our arguments use the Riemannian approximation of $\mathbb{H}^n$ combined with optimal mass transportation techniques.

Résumé

Dans cette Note, nous présentons des versions géodésiques des inégalités de Borell-Brascamp-Lieb et de Brunn-Minkowski, et des inégalités d'entropie, sur le groupe de Heisenberg $\mathbb{H}^n$. Nos démonstrations s'appuient sur l'approximation riemannienne de $\mathbb{H}^n$ et sur des techniques de transport optimal.

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1. Definitions and notations

The Heisenberg group $\mathbb{H}^n$ can be identified with its Lie algebra $\mathbb{R}^{2n+1} \simeq \mathbb{C}^n \times \mathbb{R}$ via canonical exponential coordinates, $n \in \mathbb{N}$. We denote a point in $\mathbb{H}^n$ by $x = (\xi, \eta, t) = (\zeta, t)$, where $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n$, $t \in \mathbb{R}$, and we identify the pair (ξ, η) by $\zeta \in \mathbb{C}^n$ with coordinates $\zeta_j = \xi_j + i\eta_j$ for all $j = 1, \dots, n$. The correspondence with its Lie algebra through the exponential coordinates induces the group law

$$(\zeta, t) \cdot (\zeta', t') = \left(\zeta + \zeta', t + t' + 2\mathrm{Im}\langle \zeta, \zeta' \rangle\right), \quad \forall (\zeta, t),\ (\zeta', t') \in \mathbb{C}^n \times \mathbb{R},$$

where Im denotes the imaginary part of a complex number and $\langle \zeta, \zeta' \rangle = \sum_{j=1}^{n} \zeta_j \overline{\zeta_j'}$ is the Hermitian inner product. In these coordinates the neutral element of $\mathbb{H}^n$ is $0_{\mathbb{H}^n} = (0_{\mathbb{C}^n}, 0)$ and the inverse element of (ζ, t) is $(-\zeta, -t)$. Note that $x = (\xi, \eta, t) = (\zeta, t)$ form a real coordinate system for $\mathbb{H}^n$ and the system of vector fields given as differential operators

$$X_j = \partial_{\xi_j} + 2\eta_j \partial_t, \quad Y_j = \partial_{\eta_j} - 2\xi_j \partial_t, \quad \forall j = 1, \dots n, \quad T = \partial_t,$$

form a basis for the left invariant vector fields of $\mathbb{H}^n$. The vectors $X_j, Y_j, j \in \{1, ..., n\}$ form the basis of the horizontal bundle; let d_{CC} be the associated Carnot-Carathéodory metric.

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For every $s \in (0,1)$ and $x, y \in \mathbb{H}^n$, let

$$Z_s(x,y) = \{z \in \mathbb{H}^n : d_{CC}(x,z) = s d_{CC}(x,y),\ d_{CC}(z,y) = (1-s) d_{CC}(x,y)\}$$

be the set of *s-intermediate points* between x and y. Note that $(\mathbb{H}^n, d_{CC})$ is a geodesic metric space, thus $Z_s(x,y) \neq \emptyset$ for every $x, y \in \mathbb{H}^n$. For the nonempty sets $A, B \subset \mathbb{H}^n$, let $Z_s(A,B) = \bigcup_{(x,y)\in A\times B} Z_s(x,y)$.

For $s \in (0,1)$, we introduce the *Heisenberg distortion coefficients* $\tau_s^n : [0, 2\pi] \to [0, \infty]$ defined by

$$\tau_s^n(\theta) = \begin{cases} s^{\frac{1}{2n+1}} \left(\frac{\sin \frac{\theta s}{2}}{\sin \frac{\theta}{2}} \right)^{\frac{2n-1}{2n+1}} \left(\frac{\sin \frac{\theta s}{2} - \frac{\theta s}{2} \cos \frac{\theta s}{2}}{\sin \frac{\theta}{2} - \frac{\theta}{2} \cos \frac{\theta}{2}} \right)^{\frac{1}{2n+1}} & \text{if} \quad \theta \in (0, 2\pi); \\ s^{\frac{2n+3}{2n+1}} & \text{if} \quad \theta = 0; \\ +\infty & \text{if} \quad \theta = 2\pi. \end{cases} \tag{1}$$

The function $\theta \mapsto \tau_s^n(\theta)$ is increasing on $[0, 2\pi]$ and $\tau_s^n(\theta) \to +\infty$ as $\theta \to 2\pi$; in particular,

$$\tau_s^n(\theta) \geq \tau_s^n(0) = s^{\frac{2n+3}{2n+1}} \quad \text{for every } \theta \in [0, 2\pi], s \in (0,1). \tag{2}$$

We also introduce the notation

$$\tilde{\tau}_s^n = s^{-1} \tau_s^n.$$

If $x, y \in \mathbb{H}^n$, $x \neq y$ we let $\theta(x,y) = |\theta|$ with the property that $(\chi, \theta) \in \Gamma_1^{-1}(x^{-1} \cdot y)$, where

$$\Gamma_1(\chi, \theta) = \begin{cases} \left(i \frac{e^{-i\theta} - 1}{\theta} \chi, 2|\chi|^2 \frac{\theta - \sin(\theta)}{\theta^2} \right) & \text{if} \quad \theta \neq 0; \\ (\chi, 0) & \text{if} \quad \theta = 0, \end{cases} \tag{3}$$

is the so-called *end-point map* of the Heisenberg geodesic starting from $0_{\mathbb{H}^n}$ with parameters (χ, θ), see Ambrosio and Rigot [1]. Observe, that $\theta(x,y)$ is well defined and $\theta(x,y) = \theta(y,x)$. If $x = y$ we set $\theta(x,y) = 0$.

For every $s \in (0,1)$, $p \in \mathbb{R} \cup \{\pm\infty\}$ and $a, b \geq 0$, we consider the *p-mean*

$$M_s^p(a,b) = \begin{cases} ((1-s)a^p + s b^p)^{1/p} & \text{if} \quad ab \neq 0; \\ 0 & \text{if} \quad ab = 0, \end{cases}$$

with $M_s^{+\infty}(a,b) = \max\{a,b\}$, $M_s^{-\infty}(a,b) = \min\{a,b\}$ and $M_s^0(a,b) = a^{1-s} b^s$.

2. Interpolation inequalities on $\mathbb{H}^n$

2.1. Geodesic Borell-Brascamp-Lieb inequalities on $\mathbb{H}^n$.

Theorem 2.1. (Weighted Borell-Brascamp-Lieb inequality on $\mathbb{H}^n$) *Fix $s \in (0,1)$ and $p \geq -\frac{1}{2n+1}$. Let $f, g, h : \mathbb{H}^n \to [0, \infty)$ be Lebesgue integrable functions satisfying*

$$h(z) \geq M_s^p \left(\frac{f(x)}{\tilde{\tau}_{1-s}^n(\theta(y,x))^{2n+1}}, \frac{g(y)}{\tilde{\tau}_s^n(\theta(x,y))^{2n+1}} \right) \ \forall (x,y) \in \mathbb{H}^n \times \mathbb{H}^n, z \in Z_s(x,y).$$

Then the following inequality holds:

$$\int_{\mathbb{H}^n} h \geq M_s^{\frac{p}{1+(2n+1)p}} \left(\int_{\mathbb{H}^n} f, \int_{\mathbb{H}^n} g \right).$$

By standard normalization and approximation arguments, it is enough to prove Theorem 2.1 for upper semicontinuous functions f and g with compact support such that $\int f = \int g = 1$; and lower semicontinuous functions h. We start our proof by a Riemannian approximation of the sub-Riemannian structure of $\mathbb{H}^n$ in the spirit of Ambrosio and Rigot [1]. Next, we use optimal mass transportation for compactly supported measures with densities f and g on both the approximating Riemannian manifolds and Heisenberg group $\mathbb{H}^n$. Our main ingredient is the famous interpolant Jacobi inequality on Riemannian manifolds due Cordero-Erausquin, McCann and Schmuckenschläger [3]. The last step is a careful limiting process that provides the result. Here, the limiting argument is based on an explicit computation of the volume distortion on the approximating Riemannian manifolds by using Jacobians of exponential maps and their limiting comparison with the Heisenberg distortion coefficient τ_s^n, see (1). The method of calculation of the volume distortion using Jacobians of exponential maps is inspired by the work of Juillet [6]. Further ideas/results from McCann [8] and Villani [11, 12] are deeply explored in the aforementioned arguments.

A consequence of the above result is the following:

Corollary 2.2. (Non-weighted Borell-Brascamp-Lieb inequality on $\mathbb{H}^n$) *Fix $s \in (0,1)$ and $p \geq -\frac{1}{2n+3}$. Let $f, g, h : \mathbb{H}^n \to [0,\infty)$ be Lebesgue integrable functions satisfying*

$$h(z) \geq M_s^p(f(x), g(y)) \quad for\ all\ (x,y) \in \mathbb{H}^n \times \mathbb{H}^n, z \in Z_s(x,y).$$

Then the following inequality holds:

$$\int_{\mathbb{H}^n} h \geq \frac{1}{4} M_s^{\frac{p}{1+(2n+3)p}} \left(\int_{\mathbb{H}^n} f, \int_{\mathbb{H}^n} g \right). \tag{4}$$

The proof follows by Theorem 2.1, using the monotonicity estimate (2) and basic properties of the p-mean function. We notice that the constant $\frac{1}{4}$ is sharp in (4).

2.2. Geodesic Brunn-Minkowski inequalities on $\mathbb{H}^n$.

For two measurable sets $A, B \subset \mathbb{H}^n$ we introduce the quantity

$$\Theta_{A,B} = \sup_{A_0, B_0} \inf_{(x,y) \in A_0 \times B_0} \left\{ |\theta| \in [0, 2\pi] : (\chi, \theta) \in \Gamma_1^{-1}(x^{-1} \cdot y) \right\},$$

where A_0 and B_0 are full measure subsets of A and B, respectively. The Brunn-Minkowski inequality on $\mathbb{H}^n$ is a consequence of Theorem 2.1:

Theorem 2.3. (Weighted Brunn-Minkowski inequality on $\mathbb{H}^n$) *Let $s \in (0,1)$ and A and B be two measurable sets of $\mathbb{H}^n$. Then the following geodesic Brunn-Minkowski inequality holds:*

$$\mathcal{L}^{2n+1}(Z_s(A,B))^{\frac{1}{2n+1}} \geq \tau_{1-s}^n(\Theta_{A,B}) \mathcal{L}^{2n+1}(A)^{\frac{1}{2n+1}} + \tau_s^n(\Theta_{A,B}) \mathcal{L}^{2n+1}(B)^{\frac{1}{2n+1}}. \tag{5}$$

Here we consider the outer Lebesgue measure whenever $Z_s(A,B)$ is not measurable, and the convention $+\infty \cdot 0 = 0$ for the right hand side of (5).

The value $\Theta_{A,B}$ is a typical Heisenberg quantity showing the deviation of an *essentially horizontal* position of the sets A and B. Indeed, if the sets A and B are essentially horizontal, one has $\Theta_{A,B} = 0$. However, when the sets are not essentially horizontal, the value $\Theta_{A,B}$ can be close to 2π, thus $\tau_{1-s}^n(\Theta_{A,B})$ and $\tau_s^n(\Theta_{A,B})$ can be arbitrary large. Such a situation occurs e.g. when A_r and B_r are CC-balls in $\mathbb{H}^1$ with the same small radius $r > 0$ and centers situated on the t-axis far from each other. The geodesics joining the elements of A_r and B_r largely deviate from the t-axis and $Z_s(A_r, B_r)$ becomes a large set w.r.t. A_r and B_r. In this situation we note that $\tau_s^1(\Theta_{A_r,B_r}) \geq C r^{-1/3}$ as $r \to 0$ for some $C > 0$, compensating the volume growth of $Z_s(A_r, B_r)$ on the left side of (5).

Corollary 2.4. (Non-weighted Brunn-Minkowski inequalities on $\mathbb{H}^n$) *Let $s \in (0,1)$ and A and B be two measurable sets of $\mathbb{H}^n$. Then the following inequalities hold:*

(i) $\mathcal{L}^{2n+1}(Z_s(A,B))^{\frac{1}{2n+1}} \geq (1-s)^{\frac{2n+3}{2n+1}} \mathcal{L}^{2n+1}(A)^{\frac{1}{2n+1}} + s^{\frac{2n+3}{2n+1}} \mathcal{L}^{2n+1}(B)^{\frac{1}{2n+1}}$;

(ii) $\mathcal{L}^{2n+1}(Z_s(A,B))^{\frac{1}{2n+3}} \geq \left(\frac{1}{4}\right)^{\frac{1}{2n+3}} \left((1-s)\mathcal{L}^{2n+1}(A)^{\frac{1}{2n+3}} + s\mathcal{L}^{2n+1}(B)^{\frac{1}{2n+3}}\right)$.

By shrinking one of the above sets to a point, Corollary 2.4 (i) directly implies the *measure contraction property* $\mathsf{MCP}(0, 2n+3)$ on $\mathbb{H}^n$ proved by Juillet [6, Theorem 2.3], i.e., for every $s \in [0,1]$, $x \in \mathbb{H}^n$ and measurable set $E \subset \mathbb{H}^n$,

$$\mathcal{L}^{2n+1}(Z_s(x,E)) \geq s^{2n+3}\mathcal{L}^{2n+1}(E).$$

2.3. Entropy inequality on $\mathbb{H}^n$.

By the same technique used in the proof of Theorem 2.1 we obtain an intrinsic entropy convexity on the metric measure space $(\mathbb{H}^n, d_{CC}, \mathcal{L}^{2n+1})$. This is seen as a natural sub-Riemannian counterpart of the famous curvature-dimension property $\mathsf{CD}(K,N)$ introduced by Lott and Villani [7] and Sturm [9, 10]. We recall that the usual $\mathsf{CD}(K,N)$ property does not hold on $(\mathbb{H}^n, d_{CC}, \mathcal{L}^{2n+1})$ for any $K, N \in \mathbb{R}$, see Juillet [6].

Let us recall the Rényi entropy functional on $(\mathbb{H}^n, d_{CC}, \mathcal{L}^{2n+1})$, i.e., $\mathrm{Ent}_{2n+1}(\cdot|\mathcal{L}^{2n+1}) : \mathcal{P}_2(\mathbb{H}^n, d_{CC}) \to \mathbb{R}$ given by

$$\mathrm{Ent}_{2n+1}(\mu|\mathcal{L}^{2n+1}) = -\int_{\mathbb{H}^n} \rho^{1-\frac{1}{2n+1}} \mathrm{d}\mathcal{L}^{2n+1},$$

where $\mathcal{P}_2(\mathbb{H}^n, d_{CC})$ is the usual Wasserstein space of probability measures and ρ is the density function of μ w.r.t. $\mathcal{L}^{2n+1}$.

Let μ_0 and μ_1 be two compactly supported, absolutely continuous probability measures w.r.t. $\mathcal{L}^{2n+1}$ on $\mathbb{H}^n$. According to Ambrosio and Rigot [1], there exists a unique optimal transport map $\psi : \mathbb{H}^n \to \mathbb{H}^n$ transporting μ_0 to μ_1 associated to the cost function $\frac{d_{CC}^2}{2}$. Let $s \in (0,1)$. If ψ_s is the interpolant optimal transport map associated to ψ, the push-forward measure $\mu_s = (\psi_s)_\#\mu_0$ is also absolutely continuous w.r.t. $\mathcal{L}^{2n+1}$, see Figalli and Juillet [4]. Due to Figalli and Rifford [5, Theorem 3.7], the map ψ is essentially injective and its inverse function ψ^{-1} is well defined μ_1-a.e. Note that there exists a unique pair $(\chi_x, \theta_x) \in \mathbb{C}^n \times (-2\pi, 2\pi)$ such that $x^{-1} \cdot \psi(x) = \Gamma_1(\chi_x, \theta_x)$, see (3), whenever $\psi(x)$ is not in the Heisenberg cut-locus of $x \in \mathbb{H}^n$ (and this happens for μ_0 a.e. x).

The sub-Riemannian version of the generalized entropy convexity, viewed as the intrinsic curvature-dimension condition on $(\mathbb{H}^n, d_{CC}, \mathcal{L}^{2n+1})$, is stated as follows.

Theorem 2.5. (Entropy convexity on $\mathbb{H}^n$) *Let $s \in (0,1)$, and μ_0 and μ_1 be two compactly supported, absolutely continuous probability measures w.r.t. $\mathcal{L}^{2n+1}$ on $\mathbb{H}^n$ with densities ρ_0 and ρ_1, respectively. Let $\mu_s = (\psi_s)_\#\mu_0$ be the interpolant measure between μ_0 and μ_1. Then the following entropy inequalities hold:*

$$\begin{aligned}
\mathrm{Ent}_{2n+1}(\mu_s|\mathcal{L}^{2n+1}) &\leq -\int_{\mathbb{H}^n} \tau_{1-s}^n(\theta_x)\rho_0(x)^{1-\frac{1}{2n+1}} \mathrm{d}\mathcal{L}^{2n+1}(x) - \int_{\mathbb{H}^n} \tau_s^n(\theta_{\psi^{-1}(y)})\rho_1(y)^{1-\frac{1}{2n+1}} \mathrm{d}\mathcal{L}^{2n+1}(y) \\
&\leq (1-s)^{\frac{2n+3}{2n+1}} \mathrm{Ent}_{2n+1}(\mu_0|\mathcal{L}^{2n+1}) + s^{\frac{2n+3}{2n+1}} \mathrm{Ent}_{2n+1}(\mu_1|\mathcal{L}^{2n+1}).
\end{aligned}$$

For detailed proofs, further results and comments, see paper [2].

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References

[1] L. Ambrosio, S. Rigot, Optimal mass transportation in the Heisenberg group. J. Funct. Anal. 208 (2004), no. 2, 261–301.

[2] Z. M. Balogh, A. Kristály, K. Sipos, Geometric inequalities on Heisenberg groups, 2016, arXiv preprint no. 1605.06839.

[3] D. Cordero-Erausquin, R. J. McCann, M. Schmuckenschläger, A Riemannian interpolation inequality à la Borell, Brascamp and Lieb. Invent. Math. 146 (2001), no. 2, 219–257.

[4] A. Figalli, N. Juillet, Absolute continuity of Wasserstein geodesics in the Heisenberg group. J. Funct. Anal. 255 (2008), no. 1, 133–141.

[5] A. Figalli, L. Rifford, Mass transportation on sub-Riemannian manifolds. Geom. Funct. Anal. 20 (2010), 124–159.

[6] N. Juillet, Geometric inequalities and generalized Ricci bounds in the Heisenberg group. Int. Math. Res. Notices (2009), vol. 13.

[7] J. Lott, C. Villani, Ricci curvature for metric measure spaces via optimal transport. Ann. of Math. 169 (2009), no. 3, 903–991.

[8] J. McCann, Polar factorization of maps in Riemannian manifolds. Geom. Funct. Anal. 11 (2001), 589–608.

[9] K.-T. Sturm, On the geometry of metric measure spaces. I, Acta Math. 196 (2006), no. 1, 65–131.

[10] K.-T. Sturm, On the geometry of metric measure spaces. II, Acta Math. 196 (2006), no. 1, 133–177.

[11] C. Villani, Optimal transport. Old and new. Fundamental Principles of Mathematical Sciences, 338. Springer-Verlag, Berlin (2009).

[12] C. Villani, Topics in optimal transportation. AMS, Providence, RI (2003).